

AN EXAMPLE CONCERNING HAMILTONIAN GROUPS OF SELF PRODUCT, I

SHENGDA HU AND FRANÇOIS LALONDE

ABSTRACT. We show that $(S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$, with $\lambda > 1$, is an example of symplectic manifold (X, ω) such that the $\pi_1 \text{Ham}(X \times X, \omega \oplus -\omega)$ contains extra elements than those from $\pi_1 \text{Ham}(X, \omega) \times \pi_1 \text{Ham}(X, -\omega)$.

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1. INTRODUCTION

Let (X, ω) be a compact symplectic manifold with $\dim_{\mathbb{R}} X = 2n$ and $\text{Ham}(X, \omega)$ the group of Hamiltonian diffeomorphisms. It's natural to ask how $\text{Ham}(X, \omega) \times \text{Ham}(X, -\omega)$ compares with $\text{Ham}(X \times X, \omega \oplus -\omega)$. Firstly, there is a natural injection:

$$m : \text{Ham}(X, \omega) \times \text{Ham}(X, -\omega) \hookrightarrow \text{Ham}(X \times X, \omega \oplus -\omega) : m(\phi, \psi) = (\phi, \psi)$$

Secondly, since a neighbourhood of the diagonal $\Delta \subset X \times X$ is symplectomorphic to a neighbourhood of the zero section in T^*X , it is clear that the injection m can't be surjective. On the other hand, it is not as clear how they compare homotopically. It is well known that for $(X, \omega) = (S^2, \omega_0)$, the standard 2-sphere, the two sides of m are weakly homotopic. In this article we consider the first homotopy group, and will use m to denote the induced map on π_1 as well. To save notations, we use X to denote (X, ω) and \bar{X} to denote $(X, -\omega)$.

Seidel constructed for each $\gamma \in \pi_1 \text{Ham}(X)$ an automorphism Φ_γ^X of the quantum homology ring $QH_*(X)$ as a module over itself. Let $1 = [X] \in QH_*(X)$ be the unit, then the *Seidel element* $\Psi_\gamma^X = \Phi_\gamma^X(1) \in QH_*^\times(X)$ is an invertible element.¹ The map $\Psi^X : \pi_1 \text{Ham}(X) \rightarrow QH_*^\times(X) : \Psi^X(\gamma) = \Psi_\gamma^X$ is the *Seidel homomorphism*, where QH_*^\times is a group under quantum multiplication.

In this article, we consider the example $(X, \omega) = (S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$, where ω_0 is the standard volume form on S^2 and $\lambda > 1$. We prove the following statement, using explicit computation of the Seidel elements.

Theorem 1.1. *m is not surjective on π_1 for $(X, \omega) = (S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$ with $\lambda > 1$.*

Remark 1.2. We note that, in fact, $\pi_1 \text{Ham}(X, \omega)$ already has an element S which does not come from $\pi_1 \text{Ham}$ of either of its factors. On the other hand, the factors are not symplectomorphic (after reversing one of the structures). Indeed, Gromov [1] showed that $\text{Ham}(X, \omega_0 \oplus \omega_0)$ is weakly homotopic to $SO(3) \times SO(3)$, which in turn is weakly homotopic to $\text{Ham}(S^2, \omega_0) \times \text{Ham}(S^2, \omega_0)$.

Let's start by fixing some notations. Let $\Gamma_\omega = \pi_2(M)/\sim$ where $\beta \sim \beta' \iff \omega(\beta - \beta') = c_1(TX)(\beta - \beta') = 0$. As a group, the quantum homology $QH_*(X, \omega) \cong H_*(X, \omega) \otimes \Lambda_\omega$ where Λ_ω is

¹Seidel's original construction [4] gives for each choice of a reference section an automorphism as well as an element. Here, we follow McDuff [2], choosing a canonical reference section and refer to the result as *the* Seidel morphism and element. Both will appear in the main text.

the Novikov ring

$$\Lambda_\omega = \left\{ \sum_{\beta \in \Gamma_\omega} a_\beta e^\beta \mid a_\beta \in \mathbb{R}, \forall K \in \mathbb{R}, \#\{\beta \mid a_\beta \neq 0 \text{ and } \omega(\beta) > K\} < \infty \right\}$$

graded by $\deg e^\beta = 2c_1(TX)(\beta)$. The quantum (intersection) product on $QH_*(X)$ is given by

$$a * b = \sum_{\beta \in \Gamma_\omega, c \in H_*(X, \omega)} \langle a, b, \hat{c} \rangle_\beta e^{-\beta} c$$

where $\hat{c} \in H_*(X)$ is the Poincaré dual of c under the ordinary intersection product and $\langle a, b, \hat{c} \rangle_\beta$ is the genus 0 Gromov-Witten invariant counting the number of J -holomorphic rational curves in X passing through representatives of the classes a , b and \hat{c} , representing the class β .

Next recall the effect of reversing the symplectic structure on $QH_*(X)$ and the Seidel elements. It leaves Γ_ω unchanged. Let $\tau : \pi_2(X) \rightarrow \pi_2(X) : \beta \mapsto -\beta$, it induces the ring isomorphism

$$\tau : \Lambda_\omega \rightarrow \Lambda_{-\omega} : \sum_{\beta \in \Gamma_\omega} a_\beta e^\beta \mapsto \sum_{\beta \in \Gamma_\omega} a_\beta \tau(e^\beta) = \sum_{\beta \in \Gamma_\omega} (-1)^{c_1(TX)(\beta)} a_\beta e^{-\beta}$$

The quantum homology $QH_*(X)$ and $QH_*(\overline{X})$ are isomorphic as rings via

$$\tau : QH_*(X) \rightarrow QH_*(\overline{X}) : \tau(a \otimes e^\beta) = (-1)^{n+c_1(TX)(\beta)} a \otimes e^{-\beta}$$

where $a \in H_*(X)$. Let $\gamma = [g] \in \pi_1 \text{Ham}(X)$ where $g \in \Omega_0 \text{Ham}(X, \omega)$ is a loop in $\text{Ham}(X)$ based at id and define $\tau : \pi_1 \text{Ham}(X) \rightarrow \pi_1 \text{Ham}(\overline{X})$ by $\tau(\gamma) = [g^-]$, where $g^-(t) = g(1-t)$, then the Seidel elements are related by

$$(1.1) \quad \tau(\Psi_\gamma^X) = \Psi_{\tau(\gamma)}^{\overline{X}}$$

Let (X, ω_X) and (Y, ω_Y) be compact monotone symplectic manifolds, then we have the ring isomorphism extending the Künneth isomorphism for ordinary homology:

$$(1.2) \quad QH_*(X \times Y, \omega_X \oplus \omega_Y) \cong QH_*(X, \omega_X) \otimes QH_*(Y, \omega_Y)$$

For the case under consideration, although $(X, \omega) = (S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$ is not monotone, neither is $(X, -\omega)$, the manifold $(X \times X, \omega \oplus -\omega)$ can be written as a product of monotone manifolds:

$$(X \times X, \omega \oplus -\omega) = (X_1 \times X_1, \omega_1 \oplus \lambda\omega_1)$$

where $\omega_1 = \omega_0 \oplus -\omega_0$ on $X_1 = S^2 \times S^2$. Since

$$QH_*(X_1, \omega_1) \otimes QH_*(X_1, \lambda\omega_1) \cong QH_*(S^2, \omega_0) \otimes QH_*(S^2, -\omega_0) \otimes QH_*(S^2, \lambda\omega_0) \otimes QH_*(S^2, -\lambda\omega_0)$$

it follows still that

$$QH_*(X \times X, \omega \oplus -\omega) \cong QH_*(X, \omega) \otimes QH_*(X, -\omega)$$

The Hamiltonian groups are similarly related:

$$m : \text{Ham}(X, \omega_X) \times \text{Ham}(Y, \omega_Y) \hookrightarrow \text{Ham}(X \times Y, \omega_X \oplus \omega_Y)$$

Moreover, let $\gamma_X \in \pi_1 \text{Ham}(X, \omega_X)$ and $\gamma_Y \in \pi_1 \text{Ham}(Y, \omega_Y)$ then $\gamma_{X \times Y} := m(\gamma_X, \gamma_Y) \in \pi_1 \text{Ham}(X \times Y, \omega_X \oplus \omega_Y)$. Suppose that the ring isomorphism (1.2) holds, then the respective Seidel elements are related by

$$\Psi^{X \times Y}(\gamma_{X \times Y}) = \Psi^X(\gamma_X) \otimes \Psi^Y(\gamma_Y)$$

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2. EXAMPLE: $(X, \omega) = (S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$

Let (S^2, ω_0) be the sphere with the standard symplectic structure, $X = (S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$ for some $\lambda > 0$, and $(M, \Omega) = X \times \overline{X}$. Denote the factors as \mathbb{P}_j for $j = 1, \dots, 4$. Let

$$(X', \omega') = \mathbb{P}_1 \times \mathbb{P}_4 \text{ and } (M', \Omega') = X' \times \overline{X'},$$

then M' and M are isomorphic symplectic manifolds, by switching the factors; while X' and X are isomorphic via an anti-symplectic involution on the second factor.

When $\lambda \in (1, 2]$, it's known (see for example McDuff-Tolman [3]) that $\pi_1 \text{Ham}(X)$ is generated by 3 elements: r_1 and r_2 of order 2 rotating the respective factors and an element s of infinite degree. X admits another structure of S^2 fibration over S^2 and s defines an S^1 -action on X rotating the fibers. The diagonal and the anti-diagonal are the two sections of the fibration fixed by this S^1 -action, and the weight of the action on the normal bundle of the section with bigger area is -1 .

In order to write down the Seidel elements in $QH_*(X)$ and for later convenience, we introduce a system of notations for the elements in H_* of the various spaces involved. The homology $H_*(S^2) = \mathbb{Z} \oplus 0 \oplus \mathbb{Z}$, as graded by the degree. We write $(1) \in H_2(S^2)$ and $(0) \in H_0(S^2)$ as the respective (positive) generators (with respect to the volume form ω_0). For a (positive) basis of $H_*(S^2)$ with respect to the reverse form $-\omega_0$, we write $(\overline{1}) := -(1) \in H_2(S^2)$ and $(\overline{0}) := -(0) \in H_0(S^2)$. The homology $H_*(X)$ is then generated by $(11) \in H_4(X)$, $(10), (01) \in H_2(X)$ and $(00) \in H_0(X)$, where, for example, (10) denotes the tensor $(1) \otimes (0)$. We use similar notations for the generators of $H_*(M)$, e.g. $(01\overline{0}\overline{1}) \in H_4(M)$.

The quantum homology $QH_*(S^2)$ is determined by the fact that (1) is the unit and

$$(0) * (0) = (1)e^{-(1)}$$

For $QH_*(\overline{S^2})$, we have the corresponding $\overline{\tau}$ -version:

$$(\overline{0}) * (\overline{0}) = (\overline{1})e^{-(\overline{1})} \Rightarrow (\overline{0}) * (0) = -(1)e^{(1)}$$

Note that the unit in the quantum homology $QH_*(X)$, $QH_*(X')$ and $QH_*(M)$ are respectively (11) , $(1\overline{1})$ and $(11\overline{1}\overline{1})$. We have for example

$$(01) * (10) = (00) \text{ and } (01\overline{0}\overline{1}) * (00\overline{1}\overline{1}) = (100\overline{1}\overline{1})e^{-(1000)}$$

Using these notations, let r denote the action of S^1 on S^2 fixing the poles and $\Psi_r \in QH_*(S^2)$ be the corresponding Seidel element, then

$$\Psi_r^{S^2} = (0)e^{\frac{1}{2}(1)} \text{ and } \Psi_{\tau(r)}^{\overline{S^2}} = \tau(\Psi_r^{S^2}) = (-1)^{c_1(TS^2)(\frac{1}{2}(1))}(\overline{0})e^{-\frac{1}{2}(1)} = -(\overline{0})e^{-\frac{1}{2}(1)} \in QH_*(\overline{S^2})$$

We write down the Seidel elements for R_1 and R_2 :

$$\Psi_{r_1}^X = \Psi_r^{S^2} \otimes \Psi_1^{S^2} = (01)e^{\frac{1}{2}(10)} \text{ and } \Psi_{r_2}^X = \Psi_1^{S^2} \otimes \Psi_r^{S^2} = (10)e^{\frac{1}{2}(01)}$$

Following [3], we explicitly write down the Seidel element for s :

$$\Psi_s^X = [(01) + (10)]e^{\frac{1}{2}(10) + h[(10) - (01)]} \text{ where } h = \frac{1}{6\lambda(\lambda - 1)}$$

where $\omega((10)) = 1$, $\omega((01)) = \lambda$ and $c_1((01)) = c_1((10)) = 2$. Because

$$[(01) + (10)] * [(01) - (10)] = (11) \left(e^{-(10)} - e^{-(01)} \right)$$

we see that the reversed loop s^- gives the Seidel element

$$\Psi_{s^-}^X = (\Psi_s^X)^{-1} = [(01) - (10)]e^{\frac{1}{2}(10) - h[(10) - (01)]} \left(1 + e^{(10) - (01)} + e^{2[(10) - (01)]} + \dots \right)$$

The corresponding Seidel elements in $QH_*(\overline{X})$ are:

$$\Psi_{\tau(r_1)}^{\overline{X}} = -(\overline{0}\overline{1})e^{-\frac{1}{2}(10)}, \Psi_{\tau(r_2)}^{\overline{X}} = -(\overline{1}\overline{0})e^{-\frac{1}{2}(01)} \text{ and}$$

$$\Psi_{\tau(s)}^{\overline{X}} = -[(\overline{0}\overline{1}) + (\overline{1}\overline{0})]e^{-\frac{1}{2}(10) - h[(10) - (01)]}.$$

Next we describe the Seidel elements in $QH_*(X')$. Those for r'_1 and r'_2 are:

$$\Psi_{r'_1}^{X'} = \Psi_r^{S^2} \otimes \Psi_{\tau(1)}^{\overline{S^2}} = (0\overline{1})e^{\frac{1}{2}(10)} \text{ and } \Psi_{r'_2}^{X'} = \Psi_1^{S^2} \otimes \Psi_{\tau(r)}^{\overline{S^2}} = -(1\overline{0})e^{-\frac{1}{2}(01)}.$$

To describe the Seidel elements of infinite order, we notice that (X', ω') is symplectically identified with (X, ω) by

$$(1, c) : \mathbb{CP}^1 \times \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1,$$

where c is the antipodal map. It induces on H_* the isomorphism given by

$$(1, c)_* : ((00), (01), (10), (11)) \mapsto ((00), (0\overline{1}), (10), (1\overline{1}))$$

from which can be recovered the expressions for $\Psi_{r'_1}^{X'}$ and $\Psi_{r'_2}^{X'}$ given above. Let s' be the loop conjugate to s by the map $(1, c)$ then the corresponding Seidel element is

$$\Psi_{s'}^{X'} = [(0\overline{1}) - (1\overline{0})]e^{\frac{1}{2}(10)+h[(01)+(10)]} \in QH_*(X', \omega').$$

The corresponding Seidel elements in $QH_*(\overline{X}')$ are:

$$\Psi_{\tau(r'_1)}^{\overline{X}'} = -(\overline{0}1)e^{-\frac{1}{2}(10)}, \Psi_{\tau(r'_2)}^{\overline{X}'} = (\overline{1}0)e^{\frac{1}{2}(01)} \text{ and}$$

$$\Psi_{\tau(r')}^{\overline{X}'} = -[(\overline{0}1) - (\overline{1}0)]e^{-\frac{1}{2}(10)-h[(01)+(10)]}.$$

The image of the obvious map:

$$m : \pi_1 \text{Ham}(X) \times \pi_1 \text{Ham}(\overline{X}) \rightarrow \pi_1 \text{Ham}(M)$$

is generated by the image of $\{1, r_1, r_2, s\} \times \{1, \tau(r_1), \tau(r_2), \tau(s)\}$ and the corresponding Seidel elements are given by the respective tensor products. Let m' be the corresponding map for $(X', \pm\omega')$:

$$m' : \pi_1 \text{Ham}(X') \times \pi_1 \text{Ham}(\overline{X}') \rightarrow \pi_1 \text{Ham}(M') = \pi_1 \text{Ham}(M),$$

where the last identification is by switching the factors of M' . The image of m' is generated by the image of $\{1, r'_1, r'_2, s'\} \times \{1, \tau(r'_1), \tau(r'_2), \tau(s')\}$. Simple algebraic observation together with the explicit description of the Seidel elements given above lead to

Proposition 2.1. $\text{img}(m) \neq \text{img}(m') \subset \pi_1 \text{Ham}(M, \Omega)$.

Proof: We first proceed as far as possible without using the exact form of the Seidel elements computed above. Let $S = m(s, 1)$, $T = m(1, \tau(s))$, $R_j = m(r_j, 1)$, $\overline{R}_j = m(1, \tau(r_j))$ for $j = 1, 2$ and the corresponding ones with $'$, be loops in $\text{Ham}(M, \Omega)$. Let $\Lambda := \Lambda_\Omega$ denote the Novikov ring for (M, Ω) . It's evident that

$$(2.1) \quad \begin{aligned} \Psi_S^M &\in \text{Span}_\Lambda((01\overline{11}), (10\overline{11})), \quad \Psi_T^M \in \text{Span}_\Lambda((11\overline{01}), (11\overline{10})), \text{ and} \\ \Psi_{S'}^M &\in \text{Span}_\Lambda((01\overline{11}), (11\overline{10})), \quad \Psi_{T'}^M \in \text{Span}_\Lambda((10\overline{11}), (11\overline{01})). \end{aligned}$$

More explicitly, we have the following

$$\begin{aligned} \Psi_S^M &= [(01\overline{11}) + (10\overline{11})] e^{\frac{1}{2}(1000)+h[(1000)-(0100)]} \\ \Psi_T^M &= -[(11\overline{01}) + (11\overline{10})] e^{-\frac{1}{2}(0010)-h[(0010)-(0001)]} \\ \Psi_{S'}^M &= [-(11\overline{10}) + (01\overline{11})] e^{\frac{1}{2}(1000)+h[(0001)+(1000)]} \\ \Psi_{T'}^M &= -[-(10\overline{11}) + (11\overline{01})] e^{-\frac{1}{2}(0010)-h[(0100)+(0010)]} \end{aligned}$$

We'll drop the superscripts such as X from the notation of the Seidel elements as they can be inferred from the subscripts. The Seidel elements of loops in $\text{img}(m)$ are of the form

$$\sigma := \Psi_{R_1}^{\epsilon_1} \Psi_{R_2}^{\epsilon_2} \Psi_{\overline{R}_1}^{\epsilon_3} \Psi_{\overline{R}_2}^{\epsilon_4} \Psi_S^p \Psi_T^q$$

where $\epsilon_j \in \{0, 1\}$ and $p, q \in \mathbb{Z}$. Square it we have

$$(2.2) \quad \sigma^2 = \Psi_S^{2p} \Psi_T^{2q}$$

Suppose that σ also lies in $\text{img}(m')$, then $\exists p', q' \in \mathbb{Z}$ so that

$$(2.3) \quad \sigma^2 = \Psi_S^{2p} \Psi_T^{2q} = \Psi_{S'}^{2p'} \Psi_{T'}^{2q'} = \sigma'^2$$

In the following we show that (2.3) holds iff $p = q = p' = q' = 0$.

It's easy to see from (2.1) (also see below for the first two) that

$$\begin{aligned} \Psi_S^2 \in V &:= \text{Span}_\Lambda((11\overline{11}), (00\overline{11})), \quad \Psi_T^2 \in W := \text{Span}_\Lambda((11\overline{11}), (11\overline{00})) \\ \text{and } \Psi_{S'}^2 \in V' &:= \text{Span}_\Lambda((11\overline{11}), (01\overline{10})), \quad \Psi_{T'}^2 \in W' := \text{Span}_\Lambda((11\overline{11}), (10\overline{01})). \end{aligned}$$

Notice that V, V', W and W' are closed under the quantum product $*$ and inverse (whenever exists).

Let us first assume that $p, q, p', q' \geq 0$, then σ^2 has the form:

$$(a(11\overline{11}) + b(00\overline{11})) * (c(11\overline{11}) + d(11\overline{00})) = ac(11\overline{11}) + ad(11\overline{00}) + bc(00\overline{11}) + bd(00\overline{00})$$

while σ'^2 is of the form:

$$(a'(11\overline{11}) + b'(10\overline{01})) * (c'(11\overline{11}) + d'(01\overline{10})) = a'c'(11\overline{11}) + a'd'(01\overline{10}) + b'c'(10\overline{01}) + b'd'(00\overline{00})$$

It follows that the necessary condition for (2.3) to hold is

$$(2.4) \quad ad = bc = a'd' = b'c' = 0 \in \Lambda$$

Here we need the explicit form of the Seidel elements. First we have

$$\Psi_s^2 = \left[2(00) + (11) \left(e^{-(10)} + e^{-(01)} \right) \right] e^{(10)+2h[(10)-(01)]} \in QH_*(X).$$

Now let $x = e^{-(10)}$, $y = e^{-(01)}$, $A = (00)$ and $B = (11)$, then for any integer $p > 0$

$$\Psi_s^{2p} = K^p \left(A + \frac{x+y}{2} B \right)^p, \text{ where } A^2 = Bxy, B^2 = B, AB = A \text{ and } K = 2x^{-2h-1}y^{2h}$$

We have the explicit formula

$$\Psi_s^{2p} = K^p \left(\sum_{i=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2i} \alpha^{p-2i} (xy)^i B + \sum_{i=0}^{\lfloor \frac{p-1}{2} \rfloor} \binom{p}{2i+1} \alpha^{p-2i-1} (xy)^i A \right), \text{ where } \alpha = \frac{x+y}{2}.$$

Note that

$$\tau(x) = e^{(10)} = x^{-1}, \tau(y) = e^{(01)} = y^{-1}, \tau(A) = (\overline{00}) = (00) = A \text{ and } \tau(B) = B$$

It follows that $\tau(\alpha) = (xy)^{-1}\alpha$ and $\tau(K) = 2x^{2h+1}y^{-2h} = 4K^{-1}$. Using (1.1) we get for $q > 0$

$$\Psi_{\tau(s)}^{2q} = 4^q K^{-q} \left(\sum_{i=0}^{\lfloor \frac{q}{2} \rfloor} \binom{q}{2i} \alpha^{q-2i} (xy)^{i-q} B + \sum_{i=0}^{\lfloor \frac{q-1}{2} \rfloor} \binom{q}{2i+1} \alpha^{q-2i-1} (xy)^{i+1-q} A \right),$$

Since $\Psi_S^{2p} = \Psi_s^{2p} \otimes \Psi_{\tau(1)}$ and $\Psi_T^{2q} = \Psi_1 \otimes \Psi_{\tau(s)}^{2q}$, it follows that in (2.4) $ad = bc = 0 \Rightarrow p = q = 0$, i.e. $\sigma^2 = id$. Similarly $a'd' = b'c' = 0 \Rightarrow p' = q' = 0$ and $(\sigma')^2 = id$.

The other cases of the sign combinations of p, q, p' and q' are similar. Among $p, q, -p', -q'$, there must be 2 of the same sign. Let's suppose p and $-p'$ are of the same sign, say both ≥ 0 , then instead of (2.3) we may consider

$$\Psi_S^{2p} \Psi_{S'}^{-2p'} = \Psi_T^{-2q} \Psi_{T'}^{2q'}.$$

Without using the details of the Seidel elements involved, we arrive at an equation similar to (2.4). Afterwards, explicit computation similar to the above gives $p = p' = 0$ and thus $\sigma^2 = (\sigma')^2 = id$.

It follows that, at least, all elements in the image of m of the form $pS + qT$ with p or $q \neq 0$ do not lie in the image of m' , and the proposition follows. \square

Corollary 2.2. m is not surjective on π_1 for $(X, \omega) = (S^2 \times S^2, \omega_0 \oplus \lambda\omega_0)$ with $\lambda > 1$. \square

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DEPARTMENT OF MATHEMATICS, WILFRID LAURIER UNIVERSITY, 75 UNIVERSITY AVE. WEST, WATERLOO, CANADA
E-mail address: `shu@wlu.ca`

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ DE MONTRÉAL, C.P. 6128, SUCC. CENTRE-VILLE, MONTRÉAL H3C 3J7, QUÉBEC, CANADA
E-mail address: `lalonde@dms.umontreal.ca`